REPRESENTATION THEORY OF THE RATIONAL CHEREDNIK ALGEBRAS OF TYPE $\mathbb{Z}/l\mathbb{Z}$ VIA MICROLOCAL ANALYSIS

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Abstract. Based on the methods developed in [KR], we consider microlocalization of the rational Cherednik algebra of type $\mathbb{Z}/l\mathbb{Z}$. Our goal is to construct the irreducible modules and standard modules of the rational Cherednik algebra by using the microlocalization. As a consequence, we obtain the sheaves corresponding to holonomic systems with regular singularities.

1. Introduction

The symplectic reflection algebra is a noncommutative deformation of the smash product $\mathbb{C}[V]\#\Gamma$, introduced by [EG], where V is a symplectic vector space and Γ is a finite group generated by symplectic reflections of V. Sometimes, we identify the symplectic reflection algebra with its spherical subalgebra, a noncommutative deformation of $\mathbb{C}[V]^{\Gamma}$, because these algebras are mutually Morita equivalent except for a certain choice of their parameters.

When the group Γ coincides with a complex reflection group and V coincides with $\mathfrak{h} \oplus \mathfrak{h}^*$ where $\mathfrak{h}$ is the reflection representation of Γ, the symplectic reflection algebra is sometimes called the rational Cherednik algebra. An important property of the rational Cherednik algebra is that it has a triangular decomposition similarly to complex semisimple Lie algebras. Via the triangular decomposition, we can introduce a certain subcategory of the category of modules, called the category $\mathcal{O}$. The category $\mathcal{O}$ is a highest weight category in terms of [CPS]. Its standard modules and costandard modules are studied in [GGOR]. For each irreducible $\mathbb{C}\Gamma$-module $E \in \operatorname{Irr} \mathbb{C}\Gamma$, we have a corresponding standard module $\Delta(E)$. The standard module has a unique irreducible quotient $L(E)$ and any irreducible module in the category $\mathcal{O}$ is isomorphic to $L(E)$ for a certain E. One of fundamental problems of the representation theory of the rational Cherednik algebras is to determine multiplicities $[\Delta(E) : L(F)]$ in the Grothendieck group for E, $F \in \operatorname{Irr} \mathbb{C}\Gamma$.

When the group Γ is a wreath product $(\mathbb{Z}/l\mathbb{Z}) \wr \mathfrak{S}_n$ of a cyclic group $\mathbb{Z}/l\mathbb{Z}$ and a symmetric group $\mathfrak{S}_n$, there is a close connection between the rational Cherednik algebra and a quiver variety which is a symplectic variety introduced by [Na]. After the leading work of [GS1] and [GS2], [KR] constructed a microlocalization of the rational Cherednik algebra of type $\mathfrak{S}_n$. The microlocalization is a kind of Deformation-Quantization algebra, called a W-algebra, on a quiver variety. [KR] introduced the notion of F-action of the W-algebras and established the equivalence of categories between the category of modules of the rational Cherednik algebra and the category of F-equivariant, good modules of the W-algebra. This equivalence is an analogue of the Beilinson-Bernstein correspondence for complex semisimple Lie algebras.

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In [BK], microlocalization of the rational Cherednik algebra of type $\mathbb{Z}/l\mathbb{Z}$ was studied. As an application of the microlocalization of the rational Cherednik algebras, we study the construction of the irreducible modules and the standard modules of the rational Cherednik algebra via microlocalization.

Let us describe the structure of this article.

In Section 2, we review fundamental properties of the minimal resolutions of Kleinian singularities of type A. We construct the Kleinian singularities and their resolutions X as quiver varieties of a cyclic quiver. Moreover, we see that the structure of X as a toric variety gives us an affine open covering $X = \bigcup_{i=1}^{l} X_i$ such that $X_i \simeq \mathbb{C}^2$.

In Section 3, we review the general setting of the W-algebra and construct the microlocalization $\widetilde{\mathscr{A}_c}$ of the rational Cherednik algebra A_c on X. By the theorem of [BK], we have an equivalence of categories

$$\begin{aligned} \mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c}) &\longrightarrow A_c\text{-mod} \\ \mathscr{M} &\mapsto \mathrm{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathscr{M}) \end{aligned}$$

under certain conditions on the parameter c.

At the end of Section 3.3, we describe the structure of $\widetilde{\mathscr{A}_c}|_{X_i}$ explicitly on the affine open subset X_i for $i = 1, \ldots, l$.

In Section 4, we briefly review the representation theory of the rational Cherednik algebra. Its spherical subalgebra is isomorphic to A_c. We introduce the category $\mathcal{O}(A_c)$, and review the definition of its standard modules $\Delta_c(i)$ and irreducible modules $L_c(i)$.

In Section 5.1, we construct an $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_c^{\Delta}(i)$ for $i = 1, \ldots, l$. It is also an F-equivariant, holonomic $\widetilde{\mathscr{A}_c}$-module supported on a certain Lagrangian subvariety. We show that the corresponding A_c-module $\mathrm{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(i))$ is isomorphic to the standard module $\Delta_c(i)$.

In Section 5.2, we construct an $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_c(i)$ for $i = 1, \ldots, l$. It is an F-equivariant, holonomic $\widetilde{\mathscr{A}_c}$-module supported on a certain Lagrangian subvariety. Moreover, we show that $\mathcal{L}_c(i)$ is a irreducible $\widetilde{\mathscr{A}_c}$-module for any $i = 1, \ldots, l$. At the end of Section 5.2, we determine the multiplicity $[\Delta_c(i) : L_c(j)]$ in the Grothendieck group of $\mathcal{O}(A_c)$ as a corollary of the construction of $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_c^{\Delta}(i)$ and $\mathcal{L}_c(j)$.

Finally, in Appendix A, we determine the global sections of $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_c^{\Delta}(i)$ explicitly.

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2. Quiver varieties

In this section we review the definition and fundamental properties of quiver varieties without framing, which were introduced by [Kr].

Let $Q = (I, E)$ be a cyclic quiver with vertices $I = \{I_i \mid i = 0, \ldots, l-1\}$ and arrows $E = \{\alpha_i : I_{i-1} \to I_i \mid i = 1, \ldots, l\}$. Let $\overline{Q} = (I, E \sqcup E^*)$ be a quiver with vertices I and arrows E and $E^* = \{\alpha_i^* : I_i \to I_{i-1}\}$. Throughout this paper, we regard indices of vertices and edges of Q and $\overline{Q}$ as integers modulo l, i.e. we regard

$I_{l+i} = I_i$ and $\alpha_{l+i} = \alpha_i$.

$$\overline{Q}: \quad \begin{tikzcd} & & I_0 \arrow[dll, shift left, "\alpha_1^*"] \arrow[drr, shift left, "\alpha_l^*"] & & \\ I_1 \arrow[urr, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & I_2 \arrow[l, shift right, "\alpha_2^*"'] \arrow[rrr, dotted, leftrightarrow] & & & I_{l-1} \arrow[ull, shift left, "\alpha_l"] \end{tikzcd}$$

A representation of $\overline{Q}$ with a dimension vector $\delta = (1, \dots, 1)$ is a pair $(V, (a_i, b_i)_{i=1,\dots,l})$ of an I-graded vector space $V = \bigoplus_{i\in 0}^{l-1} V_i$ such that $\dim V_i = 1$ for all $i \in I$ and linear maps $a_i : V_{i-1} \to V_i$, $b_i : V_i \to V_{i-1}$. Since $\dim V_i = 1$ for all i, we regard a_i and b_i as elements of $\mathbb{C}$. Let $GL(\delta) = \prod_{i=0}^{l-1} GL(V_i) \simeq (\mathbb{C}^*)^l$ be a reductive algebraic group acting on V. Let $G = PGL(\delta) = GL(\delta)/\mathbb{C}^*_{diag} \simeq (\mathbb{C}^*)^{l-1}$ where $\mathbb{C}^*_{diag}$ is the diagonal subgroup of $GL(\delta)$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G. We have $\mathfrak{g} = \left(\bigoplus_{i=0}^{l-1} \mathbb{C}\right)/\mathbb{C}_{diag}$ where $\mathbb{C}_{diag}$ is the diagonal of $\bigoplus_{i=0}^{l-1} \mathbb{C}$.

A representation of $\overline{Q}$:

$$\begin{tikzcd} & & \mathbb{C}^1 \arrow[dll, shift left, "b_1"] \arrow[drr, shift left, "b_l"] & & \\ \mathbb{C}^1 \arrow[urr, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & \mathbb{C}^1 \arrow[l, shift right, "b_2"'] \arrow[rrr, dotted, leftrightarrow] & & & \mathbb{C}^1 \arrow[ull, shift left, "a_l"] \end{tikzcd}$$

Fix a parameter $\theta = (\theta_0, \dots, \theta_{l-1}) \in \mathbb{Z}^l$ such that $\theta_0 + \theta_1 + \cdots + \theta_{l-1} = 0$. Note that we regard indices of θ as integers modulo l: $\theta = (\theta_i)_{i\in\mathbb{Z}/l\mathbb{Z}}$, and $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1}$ is well-defined for any i, $j \in \mathbb{Z}/l\mathbb{Z}$. We regard θ as an infinitesimal character of $\mathfrak{g}$.

A representation $(V, (a_i, b_i)_{i=1,\dots,l})$ is called θ-semistable if any I-graded subspace W of V which is stable under the action of $(a_i, b_i)_{i=1,\dots,l}$ satisfies the condition: $\sum_{i=0}^{l-1} \dim W_i \theta_i \le 0$.

Fix a parameter θ. Let $\widetilde{X}_\theta \subset \mathbb{C}^{2l}$ be the space of all θ-semistable representation,

$$\widetilde{X}_\theta = \{(a_i, b_i)_{i=1,\dots,l} \in \mathbb{C}^{2l} \mid (a_i, b_i)_{i=1,\dots,l} \text{ is } \theta\text{-semistable}\}.$$

The group G acts effectively on $\widetilde{X}$. This action is a symplectic action. Two points p, p' of X is called S-equivalent if the closures of their orbits have an intersection in $\widetilde{X}_\theta$.

Consider the following moment map with respect to the action:

$$\mu : \widetilde{X}_\theta \longrightarrow \mathfrak{g}^* \subset \mathbb{C}^l,$$
$$(a_i, b_i)_{i=1,\dots,l} \mapsto (a_{i+1}b_{i+1} - a_i b_i)_{i=0,\dots,l-1}.$$

We consider the Hamiltonian reduction with respect to the moment map μ. The subset $\mu^{-1}(0) \subset \widetilde{X}_\theta$ is stable under the action of G.

Definition 2.1. *The quiver variety of the quiver $\overline{Q}$ with the dimension vector δ and the stability parameter θ is a complex symplectic variety*

$$X_\theta = \mu^{-1}(0)/\sim_S$$

where $\sim_S$ be the S-equivalence.

We denote an S-equivalence class in X_θ containing $(a_i, b_i)_{i=1,\dots,l} \in \widetilde{X}_\theta$ by $[a_i, b_i]_{i=1,\dots,l}$.

Let us consider the case of $\theta = 0 = (0, \dots, 0)$. For $(a_i, b_i)_{i=1,\dots,l} \in \mu^{-1}(0) \subset \widetilde{X}_0 = \mathbb{C}^{2l}$, we set $\bar{a} = \sqrt[l]{a_1 \cdots a_l}$, $\bar{b} = \sqrt[l]{b_1 \cdots b_l}$ such that $\bar{a}\bar{b} = a_1 b_1$. Then, we have the following isomorphism of algebraic varieties.

$$X_0 \xrightarrow{\simeq} \mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z})$$
$$[a_i, b_i]_{i=1,\dots,l} \mapsto (\bar{a}, \bar{b})$$

Note that the image of $(\bar{a}, \bar{b})$ in $\mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z})$ is independent of the choice of root. (cf. [CS]).

Proposition 2.2 ([CS], [Kr])**.** *If a stability parameter $\theta = (\theta_i)_{i=0,\dots,l-1}$ satisfies $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1} \neq 0$ for all i, j ($i \neq j$), X_θ is nonsingular and we have a minimal resolution of Kleinian singularities of type A_{l-1}:*

$$\pi_\theta : X_\theta \longrightarrow X_0 \simeq \mathbb{C}^2/(\mathbb{Z}/l\mathbb{Z}).$$

In the rest of this paper, we fix a stability parameter θ satisfying the condition of Proposition 2.2. We denote X_θ by X, π_θ by π, ... for simplicity.

One of the fundamental properties of X is that it is a toric variety with respect to the following action of a 2-dimensional torus $\mathbb{T}^2 = (\mathbb{C}^*)^2$:

$$(q_1, q_2)[a_i, b_i]_{i=1,\dots,l} = [q_1 a_i, q_2 b_i]_{i=1,\dots,l}$$

for $(q_1, q_2) \in \mathbb{T}^2$ and $[a_i, b_i]_{i=1,\dots,l} \in X$. The following facts are easy to obtain from the general theory of toric varieties. Refer to [Ku, Section 2] for proofs of these facts, or to [Fu] for the general theory of toric varieties.

The variety X has l $\mathbb{T}$-fixed points $p'_1, \dots, p'_l$ where $p'_i = [a_j, b_j]_{j=1,\dots,l}$ is given as follows:

$$\begin{aligned} a_i = 0, \quad b_i &= 0, \\ a_j = 0, \quad b_j \neq 0 \quad &\text{if } \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} < 0, \\ a_j \neq 0, \quad b_j = 0 \quad &\text{if } \theta_i + \theta_{i+1} + \cdots + \theta_{j-1} > 0. \end{aligned}$$

Note that we have $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1} \neq 0$ for all $i \neq j$ by the condition of the stability parameter θ.

Define an ordering $\rhd$ on the set of indices $\Lambda = \{1, \dots, l\}$ by

$$i \rhd j \Longleftrightarrow \theta_i + \cdots + \theta_{j-1} < 0.$$

By the condition of the stability parameter θ, the ordering $\rhd$ is a total ordering. Let $\eta_1, \dots, \eta_l$ be the indices in Λ such that

$$\eta_1 \rhd \eta_2 \rhd \cdots \rhd \eta_l. \tag{1}$$

Remark 2.3. *Note that the order of the numbering $\eta_1, \dots, \eta_i$ is reversed from the one of* [Ku].

Set $p_i = p'_{\eta_i}$ for $i = 1, \dots, l$. The explicit description of the point p_i is given as follows.

Lemma 2.4. *For $i = 1, \dots, l$, the fixed point $p_i = p'_{\eta_i} = [a_j, b_j]_{j=1,\dots,l}$ is given by*

$$\begin{aligned} a_{\eta_i} = 0, \quad b_{\eta_i} &= 0, \\ a_{\eta_j} = 0, \quad b_{\eta_j} \neq 0 \quad &\textit{for } j > i, \\ a_{\eta_j} \neq 0, \quad b_{\eta_j} = 0 \quad &\textit{for } j \leq i. \end{aligned}$$

Let us consider a Lagrangian subvariety $\pi^{-1}(\{\bar{a} = 0 \text{ or } \bar{b} = 0\})$. This subvariety has $(l+1)$ irreducible components $D_0, D_1, \dots, D_l$ such that $D_0, D_l \simeq \mathbb{C}^1$, $D_i \simeq \mathbb{P}^1$ for $1 \leq i \leq l-1$ and p_i is a unique intersection of D_{i-1} and D_i. We can describe D_i explicitly as follows.

Lemma 2.5. *For $i = 1, \dots, l$, the $\mathbb{T}$-divisor D_i is given by*

$$D_i = \overline{\left\{ [a_j, b_j]_{j=1,\dots,l} \;\middle|\; \begin{array}{ll} a_{\eta_j} = 0,\ b_{\eta_j} \neq 0 & \textit{for } j > i, \\ a_{\eta_j} \neq 0,\ b_{\eta_j} = 0 & \textit{for } j \leq i. \end{array} \right\}}.$$

Similarly, D_0 is given by

$$D_0 = \overline{\{[a_j, b_j]_{j=1,\dots,l} \mid b_j \neq 0\}}.$$

The description as a toric variety gives us the following affine open covering of X,

$$X = \bigcup_{i=1}^{l} X_i, \qquad X_i = \left\{ [a_j, b_j]_{j=1,\dots,l} \;\middle|\; \begin{array}{ll} a_{\eta_j} \neq 0 & \text{for } j < i \\ b_{\eta_j} \neq 0 & \text{for } j > i \end{array} \right\}.$$

We introduce coordinate functions $\bar{x}_i$ (resp. $\bar{y}_i$) for $i = 1, \dots, l$ on $\widetilde{X} \subset \mathbb{C}^{2l}$ defined by $\bar{x}_i((a_j, b_j)_{j=1,\dots,l}) = a_i$ (resp. $\bar{y}_i((a_j, b_j)_{j=1,\dots,l}) = b_i$). For $i = 1, \dots, l$, let R_i be the following subring of $\mathbb{C}(\bar{x}_1, \dots, \bar{x}_l, \bar{y}_1, \dots, \bar{y}_l)$ which is isomorphic to a polynomial ring in 2-variables:

$$R_i = \mathbb{C}[\bar{f}_i, \bar{g}_i]$$

where

$$\bar{f}_i = \frac{\bar{x}_{\eta_1}\bar{x}_{\eta_2}\dots\bar{x}_{\eta_i}}{\bar{y}_{\eta_{i+1}}\bar{y}_{\eta_{i+2}}\dots\bar{y}_{\eta_l}}, \quad \bar{g}_i = \frac{\bar{y}_{\eta_i}\bar{y}_{\eta_{i+1}}\dots\bar{y}_{\eta_l}}{\bar{x}_{\eta_1}\bar{x}_{\eta_2}\dots\bar{x}_{\eta_{i-1}}} \in \mathbb{C}(\bar{x}_1, \dots, \bar{x}_l, \bar{y}_1, \dots, \bar{y}_l)$$

Then we have

$$X_i = \operatorname{Spec} R_i \simeq \mathbb{C}^2 = T^*\mathbb{C}^1.$$

Note that $\bar{f}_i\bar{g}_{i+1} = 1$ on $X_i \cap X_{i+1}$.

For $i = 1, \dots, l$, the fixed point p_i belongs to X_i. For $i = 1, \dots, l-1$, we have $D_i \simeq \mathbb{P}^1 \subset X_i \cup X_{i+1}$ and there is an isomorphism $X_i \cup X_{i+1} \simeq T^*\mathbb{P}^1$.

3. W-algebra

In this section, we recall the definition of W-algebras ($\hbar$-localized DQ-algebras), and construct a W-algebra on X by quantum Hamiltonian reduction. We introduce a quantized symplectic coordinates of the W-algebra on X. In the rest of the paper, we consider complex manifolds equipped with the analytic topology. For a manifold M, we denote the sheaf of holomorphic functions on M by $\mathcal{O}_M$.

3.1. **Definition of W-algebras.** Let $\hbar$ be an indeterminant. Given $m \in \mathbb{Z}$, let $\mathscr{W}_{T^*\mathbb{C}^n}(m)$ be a sheaf of formal series $\sum_{k \geq -m} \hbar^k a_k$ ($a_k \in \mathcal{O}_{T^*\mathbb{C}^n}$) on the cotangent bundle $T^*\mathbb{C}^n$ of $\mathbb{C}^n$. We set $\mathscr{W}_{T^*\mathbb{C}^n} = \bigcup_m \mathscr{W}_{T^*\mathbb{C}^n}(m)$. We define a noncommutative $\mathbb{C}((\hbar))$-algebra structure on $\mathscr{W}_{T^*\mathbb{C}^n}$ by

$$f \circ g = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \hbar^{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha f \cdot \partial_x^\alpha g$$

where, for a multi-power $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let X be a complex symplectic manifold with symplectic form ω. A W-algebra on X is a sheaf of $\mathbb{C}((\hbar))$-algebras $\mathscr{W}$ such that for any point $x \in X$, there is an open neighbourhood U of x, a symplectic map $\phi : U \to T^*\mathbb{C}^n$, and a $\mathbb{C}((\hbar))$-algebra isomorphism $\psi : \mathscr{W}|_U \xrightarrow{\sim} \phi^{-1}\mathscr{W}_{T^*\mathbb{C}^n}$.

The following fundamental properties of a W-algebra $\mathscr{W}$ are listed in [KR].

1. The algebra $\mathscr{W}$ is a coherent and noetherian algebra.
2. $\mathscr{W}$ contains a canonical $\mathbb{C}[[\hbar]]$-subalgebra $\mathscr{W}(0)$ which is locally isomorphic to $\mathscr{W}_{T^*\mathbb{C}^n}(0)$ (via the maps ψ). We set $\mathscr{W}(m) = \hbar^{-m}\mathscr{W}(0)$.
3. We have a canonical $\mathbb{C}$-algebra isomorphism $\mathscr{W}(0)/\mathscr{W}(-1) \xrightarrow{\sim} \mathcal{O}_X$ (coming from the canonical isomorphism via the maps ψ). The corresponding morphism $\sigma_m : \mathscr{W}(m) \to \hbar^{-m}\mathcal{O}_X$ is called the *symbol map*.
4. We have
$$\sigma_0(\hbar^{-1}[f, g]) = \{\sigma_0(f), \sigma_0(g)\}$$
for any f, $g \in \mathscr{W}(0)$. Here $\{\,\bullet\,,\,\bullet\,\}$ is the Poisson bracket.
5. The canonical map $\mathscr{W}(0) \to \varprojlim_{m\to\infty} \mathscr{W}(0)/\mathscr{W}(-m)$ is an isomorphism.

(6) A section a of $\mathscr{W}(0)$ is invertible in $\mathscr{W}(0)$ if and only if $\sigma_0(a)$ is invertible in $\mathcal{O}_X$.

(7) Given ϕ, a $\mathbb{C}((\hbar))$-algebra automorphism of $\mathscr{W}$, we can find locally an invertible section a of $\mathscr{W}(0)$ such that $\phi = \mathrm{Ad}(a)$. Moreover a is unique up to a scalar multiple. In other words, we have canonical isomorphisms

$$\begin{tikzcd} \mathscr{W}(0)^\times/\mathbb{C}[[\hbar]]^\times \arrow[r, "\sim", "\mathrm{Ad}"'] \arrow[d, "\sim"'] & \mathrm{Aut}(\mathscr{W}(0)) \arrow[d, "\sim"] \\ \mathscr{W}^\times/\mathbb{C}((\hbar))^\times \arrow[r, "\sim", "\mathrm{Ad}"'] & \mathrm{Aut}(\mathscr{W}). \end{tikzcd}$$

(8) Let v be a $\mathbb{C}((\hbar))$-linear filtration-preserving derivation of $\mathscr{W}$. Then there exists locally a section a of $\mathscr{W}(1)$ such that $v = \mathrm{ad}(a)$. Moreover a is unique up to a scalar. In other words, we have an isomorphism

$$\mathscr{W}(1)/\hbar^{-1}\mathbb{C}[[\hbar]] \xrightarrow[\mathrm{ad}]{\sim} \mathrm{Der}_{\mathrm{filtered}}(\mathscr{W}).$$

(9) If $\mathscr{W}$ is a W-algebra, then its opposite ring $\mathscr{W}^{\mathrm{opp}}$ is a W-algebra on X^{opp} where X^{opp} is a symplectic manifold with symplectic form $-\omega$.

A tuple $(f_1, \dots, f_n; g_1, \dots, g_n)$ of elements $f_i,\ g_i \in \mathscr{W}(0)$ is called quantized symplectic coordinates of $\mathscr{W}$ if they satisfy $[f_i, f_j] = [g_i, g_j] = 0$ and $[g_i, f_j] = \hbar\delta_{ij}$.

For a $\mathscr{W}$-module $\mathscr{M}$, a $\mathscr{W}(0)$-lattice of $\mathscr{M}$ is a coherent $\mathscr{W}(0)$-submodule $\mathscr{M}(0)$ such that the canonical homomorphism $\mathscr{W} \otimes_{\mathscr{W}(0)} \mathscr{M}(0) \to \mathscr{M}$ is an isomorphism. We say that a $\mathscr{W}$-module $\mathscr{M}$ is good if, for any relatively compact open subset U of X, there is a coherent $\mathscr{W}(0)|_U$-lattice of $\mathscr{M}|_U$. We denote the category of $\mathscr{W}$-modules by $\mathrm{Mod}(\mathscr{W})$ and the full subcategory of good $\mathscr{W}$-modules by $\mathrm{Mod}^{good}(\mathscr{W})$. Then, $\mathrm{Mod}^{good}(\mathscr{W})$ is an abelian subcategory of $\mathrm{Mod}(\mathscr{W})$.

Next, we review the notion of F-actions.

Let X be a symplectic manifold with the action of $\mathbb{G}_m$: $\mathbb{C}^* \ni t \mapsto T_t \in \mathrm{Aut}(X)$. We assume there exists a positive integer $m \in \mathbb{Z}_{>0}$ such that $T_t^*\omega = t^m\omega$ for all $t \in \mathbb{C}^*$.

An F-action with exponent m on $\mathscr{W}$ is an action of $\mathbb{G}_m$ on the $\mathbb{C}$-algebra $\mathscr{W}$, $\mathscr{F}_t : T_t^{-1}\mathscr{W} \xrightarrow{\sim} \mathscr{W}$ for $t \in \mathbb{C}^*$ such that $\mathscr{F}_t(\hbar) = t^m\hbar$ and $\mathscr{F}_t(f)$ depends holomorphically on t for any $f \in \mathscr{W}$.

An F-action with exponent m on $\mathscr{W}$ extends to an F-action with exponent 1 on $\mathscr{W}[\hbar^{1/m}] = \mathbb{C}((\hbar^{1/m})) \otimes_{\mathbb{C}((\hbar))} \mathscr{W}$ given by $\mathscr{F}_t(\hbar^{1/m}) = t^1\hbar^{1/m}$.

Definition 3.1. *A $\mathscr{W}[\hbar^{1/m}]$-module with an F-action is a $\mathbb{G}_m$-equivariant $\mathscr{W}[\hbar^{1/m}]$-module: i.e. there exist isomorphisms $\mathscr{F}_t : T_t^{-1}\mathscr{M} \xrightarrow{\sim} \mathscr{M}$ for $t \in \mathbb{C}^*$, and we assume that*

(1) *$\mathscr{F}_t(u)$ depends holomorphically on t for any $u \in \mathscr{M}$;*
(2) *$\mathscr{F}_t(fu) = \mathscr{F}_t(f)\mathscr{F}_t(u)$ for $f \in \mathscr{W}[\hbar^{1/m}]$ and $u \in \mathscr{M}$; and*
(3) *$\mathscr{F}_t \circ \mathscr{F}_{t'} = \mathscr{F}_{tt'}$ for $t,\ t' \in \mathbb{C}^*$.*

We denote by $\mathrm{Mod}_F(\mathscr{W}[\hbar^{1/m}])$ the category of $\mathscr{W}[\hbar^{1/m}]$-module with F-action, and by $\mathrm{Mod}_F^{good}(\mathscr{W}[\hbar^{1/m}])$ its full subcategory of good $\mathscr{W}[\hbar^{1/m}]$-modules with F-action. These are $\mathbb{C}$-linear abelian categories.

3.2. **Holonomic $\mathscr{W}$-modules.**

In this section, we review the notion of holonomic $\mathscr{W}$-module introduced in [KS2]. The following proposition is due to [KS].

Proposition 3.2 ([KS], Prop. 2.3.17)**.** *For a good $\mathscr{W}$-module $\mathscr{M}$, $\mathrm{Supp}\,\mathscr{M}$ is involutive with respect to the Poisson bracket of X. In particular, we have $\dim \mathrm{Supp}\,\mathscr{M} \geq \dim X/2$.*

Definition 3.3 ([KS2]). *We call a good $\mathscr{W}$-module $\mathscr{M}$ holonomic if* $\operatorname{Supp}\mathscr{M}$ *is a Lagrangian subvariety of X, i.e.,* $\dim\operatorname{Supp}\mathscr{M} = \dim X/2$.

Proposition 3.4. *The category of all holonomic $\mathscr{W}$-modules* $\mathrm{Mod}^{hol}(\mathscr{W})$ *is an abelian subcategory of* $\mathrm{Mod}^{good}(\mathscr{W})$.

The following lemma is obvious.

Lemma 3.5. *Let $\mathscr{M}$ be a good $\mathscr{W}$-module such that* $\operatorname{Supp}\mathscr{M}$ *is the disjoint union of subsets Z_1 and Z_2. Then, there exist two submodules $\mathscr{N}_1$, $\mathscr{N}_2$ of $\mathscr{M}$, and we have $\mathscr{M} = \mathscr{N}_1 \oplus \mathscr{N}_2$.*

Proof. Define the submodules

$$\mathscr{N}_i = \{m \in \mathscr{M} \mid \operatorname{Supp} m \subset Z_i\}$$

for $i = 1, 2$. Then the claim of lemma immediately follows. □

In the present paper, we consider the case $\dim X = 2$.

Let $\bar{x}$, $\bar{\xi} \in \mathcal{O}_{T^*\mathbb{C}^1}$ be coordinate functions on $T^*\mathbb{C}^1$ defined by $\bar{x}((a,b)) = a$, $\bar{\xi}((a,b)) = b$ for $(a,b) \in T^*\mathbb{C}^1$. Let x, $\xi \in \mathscr{W}_{T^*\mathbb{C}^1}(0)$ be the standard quantized symplectic coordinates. That is we have $[\xi, x] = \hbar$ and $\sigma_0(x) = \bar{x}$, $\sigma_0(\xi) = \bar{\xi}$.

For $\lambda \in \mathbb{C}$, let $\mathscr{M}_\lambda$ be a $\mathscr{W}_{T^*\mathbb{C}^1}$-module defined by

$$\mathscr{M}_\lambda = \mathscr{W}_{T^*\mathbb{C}^1}/\mathscr{W}_{T^*\mathbb{C}^1}(x\xi - \hbar\lambda).$$

Then, $\mathscr{M}_\lambda$ is a holonomic $\mathscr{W}_{T^*\mathbb{C}^1}$-module supported on $\{\bar{x}\bar{\xi} = 0\} \subset T^*\mathbb{C}^1$. Let v_λ be the image of the constant section $1 \in \mathscr{W}_{T^*\mathbb{C}^1}$ in $\mathscr{M}_\lambda$.

Lemma 3.6. *For $m \in \mathbb{Z}$, we have the following isomorphism of $\mathscr{W}_{T^*\mathbb{C}^1}|_{\{\bar{x}\neq 0\}}$-modules:*

$$\mathscr{M}_\lambda|_{\{\bar{x}\neq 0\}} \longrightarrow \mathscr{M}_{\lambda+m}|_{\{\bar{x}\neq 0\}},$$
$$v_\lambda \mapsto x^{-m} v_{\lambda+m}.$$

Obviously, the inverse homomorphism is given by $v_{\lambda+m} \mapsto x^m v_\lambda$.

A similar proposition holds globally on $T^*\mathbb{C}^1$. It is an analogue of a well-known fact on regular holonomic $\mathcal{D}_{\mathbb{C}^1}$-modules.

Proposition 3.7. *For any $\lambda \neq -1$, we have an isomorphism of $\mathscr{W}_{T^*\mathbb{C}^1}$-modules, $\mathscr{M}_\lambda \simeq \mathscr{M}_{\lambda+1}$.*

Proof. Define homomorphisms of $\mathscr{W}_{T^*\mathbb{C}^1}$-modules

$$\phi : \mathscr{M}_\lambda \longrightarrow \mathscr{M}_{\lambda+1}, \quad v_\lambda \mapsto \hbar^{-1}\xi v_{\lambda+1},$$

and

$$\psi : \mathscr{M}_{\lambda+1} \longrightarrow \mathscr{M}_\lambda, \quad v_{\lambda+1} \mapsto \frac{1}{\lambda+1} x v_\lambda.$$

These homomorphisms are mutually inverse, i.e.

$$\phi \circ \psi(v_{\lambda+1}) = \phi\left(\frac{1}{\lambda+1} x v_\lambda\right) = \frac{\hbar^{-1}}{\lambda+1}(x \circ \xi) v_{\lambda+1} = v_{\lambda+1},$$

and

$$\psi \circ \phi(v_\lambda) = \psi(\hbar^{-1}\xi v_\lambda) = \frac{\hbar^{-1}}{\lambda+1}(\xi \circ x) v_\lambda = v_\lambda.$$

Therefore $\mathscr{M}_\lambda$ and $\mathscr{M}_{\lambda+1}$ are isomorphic. □

The following proposition is essential for the microlocal analysis of holonomic $\mathscr{W}_{T^*\mathbb{C}^1}$-modules. This is an analogue of a consequence of the classification theorem of simple holonomic system (cf. [Ka, Proposition 8.36]).

Proposition 3.8. *Set* $Z_1 = \{x = 0\}$ *and* $Z_2 = \{\xi = 0\}$*. Note that* $\operatorname{Supp} \mathscr{M}_\lambda = Z_1 \cup Z_2$*. Then,*

(1) *For* $\lambda \notin \mathbb{Z}$*,* $\mathscr{M}_\lambda$ *is an irreducible* $\mathscr{W}_{T^*\mathbb{C}^1}$*-module.*
(2) *For* $\lambda \in \mathbb{Z}_{\geq 0}$*, there exist a* $\mathscr{W}_{T^*\mathbb{C}^1}$*-submodule* $\mathscr{N}$ *of* $\mathscr{M}_\lambda$ *supported on* Z_1*, and* $\operatorname{Supp} \mathscr{M}_\lambda/\mathscr{N} = Z_2$ *on a neighborhood of* $\{x = \xi = 0\}$*.*
(3) *For* $\lambda \in \mathbb{Z}_{<0}$*, there exist a* $\mathscr{W}_{T^*\mathbb{C}^1}$*-submodule* $\mathscr{N}$ *of* $\mathscr{M}_\lambda$ *supported on* Z_2*, and* $\operatorname{Supp} \mathscr{M}_\lambda/\mathscr{N} = Z_1$ *on a neighborhood of* $\{x = \xi = 0\}$*.*

Proof. The proof of this proposition is similar to one of [Ka, Proposition 8.36]. □

3.3. **W-algebra $\widetilde{\mathscr{A}_c}$ on the quiver variety X.** We denote the restriction of the canonical W-algebra $\mathscr{W}_{T^*\mathbb{C}^l}$ to $\widetilde{X} \subset T^*\mathbb{C}^l$ by $\mathscr{W}_{\widetilde{X}}$. Let $(x_1, \ldots, x_l; y_1, \ldots, y_l)$ (x_i, $y_i \in \mathscr{W}_{T^*\mathbb{C}^l}$) be the standard quantized symplectic coordinates: $[x_i, x_j] = [y_i, y_j] = 0$ and $[y_i, x_j] = \delta_{ij}\hbar$ for all i, j. The action of the reductive group G on $\widetilde{X}$ induces an action on $\mathscr{W}_{\widetilde{X}}$. We define the following homomorphism $\mu_{\widetilde{X}}$ of Lie algebras

$$\begin{aligned} \mu_{\widetilde{X}} : \mathfrak{g} &\longrightarrow \mathscr{W}_{\widetilde{X}}(1), \\ A_i &\mapsto \hbar^{-1}(x_{i+1}y_{i+1} - x_i y_i). \end{aligned}$$

We call $\mu_{\widetilde{X}}$ a quantum moment map with respect to the action of G. Fix a parameter $c_0, \ldots, c_{l-1} \in \mathbb{C}$ such that $c_0 + \cdots + c_{l-1} = 0$. We define a $\mathscr{W}_{\widetilde{X}}$-module $\mathscr{L}_c$ by

$$\mathscr{L}_c = \mathscr{W}_{\widetilde{X}} \Big/ \sum_{i=0}^{l-1} \mathscr{W}_{\widetilde{X}}(\mu_{\widetilde{X}}(A_i) + c_i) = \mathscr{W}_{\widetilde{X}} \Big/ \sum_{i=0}^{l-1} \mathscr{W}_{\widetilde{X}}(x_{i+1}y_{i+1} - x_i y_i + \hbar c_i).$$

The $\mathscr{W}_{\widetilde{X}}$-module $\mathscr{L}_c$ is a good $\mathscr{W}_{\widetilde{X}}$-module with a $\mathscr{W}_{\widetilde{X}}(0)$-lattice

$$\mathscr{L}_c(0) \underset{\text{def}}{=} \mathscr{W}_{\widetilde{X}}(0) \Big/ \sum_{i=0}^{l-1} \mathscr{W}_{\widetilde{X}}(0)(x_{i+1}y_{i+1} - x_i y_i + \hbar c_i).$$

Define a sheaf of algebras on X,

$$\mathscr{A}_c = \left(p_* \mathscr{E}nd_{\mathscr{W}_{\widetilde{X}}}(\mathscr{L}_c)^G\right)^{\text{opp}}$$

where $p : \mu^{-1}(0) \to X$ is the projection. By [KR], $\mathscr{A}_c$ is a W-algebra on X. Set

$$\mathscr{A}_c(0) = \left(p_* \mathscr{E}nd_{\mathscr{W}_{\widetilde{X}}(0)}(\mathscr{L}_c(0))^G\right)^{\text{opp}}.$$

Then, $\mathscr{A}_c(0)$ is a canonical $\mathbb{C}[[\hbar]]$-subalgebra of $\mathscr{A}_c$.

Let us give an F-action on $\mathscr{W}_{\widetilde{X}}$ by $\mathscr{F}_t(x_i) = tx_i$, $\mathscr{F}_t(y_i) = ty_i$, and $\mathscr{F}_t(\hbar) = t^2\hbar$ for $t \in \mathbb{C}^*$. The corresponding $\mathbb{G}_m$-action on $\widetilde{X}$ is given by $T_t((a_i, b_i)_{i=1,\ldots,l}) = (ta_i, tb_i)_{i=1,\ldots,l}$. This action induces a $\mathbb{G}_m$-action on the quiver variety X which coincides with the action induced from the toric $\mathbb{T}^2$-action on X.

The F-actions on $\mathscr{W}_{\widetilde{X}}$ induce an F-action with exponent 2 on $\mathscr{A}_c$. We set $\widetilde{\mathscr{A}_c} = \mathscr{A}_c[\hbar^{1/2}]$ and $\widetilde{\mathscr{A}_c}(0) = \mathscr{A}_c(0)[\hbar^{1/2}]$.

In [BK], we have the following W-affinity of the algebra $\widetilde{\mathscr{A}_c}$.

Theorem 3.9 ([BK])**.** *Let* $A_c = \left(\operatorname{End}_{\operatorname{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})\right)^{\text{opp}}$*.*

Assume that $c_i + c_{i+1} + \cdots + c_{j-1} \in \mathbb{Z}_{\geq 0}$ *implies* $\theta_i + \theta_{i+1} + \cdots + \theta_{j-1} < 0$*, i.e.* $i \rhd j$*. Then we have the following equivalence of categories:*

$$\begin{aligned} \operatorname{Mod}_F^{good}(\widetilde{\mathscr{A}_c}) &\simeq A_c\text{-mod}, \\ \mathscr{M} &\mapsto \operatorname{Hom}_{\operatorname{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathscr{M}). \end{aligned}$$

Its quasi-inverse functor is given by $M \mapsto \widetilde{\mathscr{A}_c} \otimes_{A_c} M$*.*

Remark 3.10. *As we see in Section 4, the algebra A_c is isomorphic to the spherical subalgebra of the rational Cherednik algebra of type $\mathbb{Z}/l\mathbb{Z}$.*

In the rest of this paper, we assume the assumptions of Theorem 3.9.

Let $\mathbf{1}_c$ be the image of the constant section $1 \in \mathscr{W}_{\widetilde{X}}$ in $\mathscr{L}_c$. For a G-invariant section $f \in \mathscr{W}_{\widetilde{X}}$, a G-invariant endomorphism of $\mathscr{L}_c$ is uniquely defined by the right multiplication $g\mathbf{1}_c \mapsto gf\mathbf{1}_c \in \mathscr{E}nd_{\mathscr{W}_{\widetilde{X}}}(\mathscr{L}_c)$ for $g \in \mathscr{W}_{\widetilde{X}}$. By abuse of notation, we denote the image of the above G-invariant endomorphism in $\widetilde{\mathscr{A}_c}$ by the same symbol f.

Consider global sections $x_1 \cdots x_l$, $y_1 \cdots y_l$, $x_i y_i$ of $\widetilde{\mathscr{A}_c}$ for $i = 1, \ldots, l$. Although these sections are not F-invariant, the global sections $\hbar^{-l/2}x_1 \cdots x_l$, $\hbar^{-l/2}y_1 \cdots y_l$, $\hbar^{-1}x_i y_i$ are elements of $A_c = \left(\mathrm{End}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c})\right)^{\mathrm{opp}}$. In $\widetilde{\mathscr{A}_c}$, we have relations

$$x_{i+1}y_{i+1} - x_i y_i + \hbar c_i = 0$$

for $i = 1, \ldots, l$.

Next, we consider the local structure of the W-algebra $\widetilde{\mathscr{A}_c}$ on the affine open subset X_i for $i = 1, \ldots, l$. Set $\widetilde{\mathscr{A}_{c,i}} = \widetilde{\mathscr{A}_c}|_{X_i}$. Recall the indices $\eta_1, \ldots, \eta_l$ in (1). We define local sections of $\widetilde{\mathscr{A}_c}$

$$f_i = (x_{\eta_1} \cdots x_{\eta_i}) \circ (y_{\eta_{i+1}} \cdots y_{\eta_l})^{-1}, \quad g_i = (y_{\eta_i} \cdots y_{\eta_l}) \circ (x_{\eta_1} \cdots x_{\eta_{i-1}})^{-1},$$

on X_i. We have $f_i \circ g_i = x_{\eta_i}y_{\eta_i}$ and $g_i \circ f_i = x_{\eta_i}y_{\eta_i} + \hbar$. Thus, for $i = 1, \ldots, l$, $\widetilde{\mathscr{A}_{c,i}}$ is isomorphic to $\mathscr{W}_{T^*\mathbb{C}^1}$ by $x \mapsto f_i$, $\xi \mapsto g_i$. That is, $(f_i; g_i)$ is a quantized symplectic coordinate of $\widetilde{\mathscr{A}_{c,i}}$. We have

$$y_1 \cdots y_l = g_i \circ (x_{\eta_1}y_{\eta_1}) \circ \cdots \circ (x_{\eta_{i-1}}y_{\eta_{i-1}}) \quad \text{on } X_i \tag{2}$$

We also have $g_{i+1} \circ f_i = f_i \circ g_{i+1} = 1$ in $\widetilde{\mathscr{A}_c}|_{X_i \cap X_{i+1}}$. Sometimes, we denote the section $g_{i+1}|_{X_i \cap X_{i+1}}$ by f_i^{-1}.

For $i = 1, \ldots, l-1$, we set

$$\tilde{c}_i = c_{\eta_i} + c_{\eta_i+1} + \cdots + c_{\eta_{i+1}-1}.$$

Under the assumption of Theorem 3.9, we have $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_{j-1} \notin \mathbb{Z}_{\leq 0}$ for $1 \leq i < j \leq l$. Then, we have

$$x_{\eta_{i+1}}y_{\eta_{i+1}} - x_{\eta_i}y_{\eta_i} + \hbar\tilde{c}_i = 0. \tag{3}$$

4. The rational Cherednik algebra and category $\mathcal{O}$

In this section, we review the definition and fundamental facts of the rational Cherednik algebra of type $\mathbb{Z}/l\mathbb{Z}$ and the category $\mathcal{O}$.

Let $\mathbb{Z}/l\mathbb{Z} = \langle\gamma\rangle$ be a cyclic group with an action on $\mathbb{C}$ given by $\gamma \mapsto \zeta = \exp(2\pi\sqrt{-1}/l)$. Let $D(\mathbb{C}^*)$ be the algebra of algebraic differential operators on $\mathbb{C}^*$. Let z be a standard coordinate function on $\mathbb{C}$. Then, we have $\mathbb{C} = \mathrm{Spec}[z]$ and $\mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[z, z^{-1}]$. The algebra $D(\mathbb{C}^*)$ is generated by $z^{\pm}$ and d/dz.

The action of $\mathbb{Z}/l\mathbb{Z}$ on $\mathbb{C}$ induces an action of $D(\mathbb{C}^*)$ given by $\gamma(z) = \zeta^{-1}z$, $\gamma(d/dz) = \zeta d/dz$. We denote the smash product of $D(\mathbb{C}^*)$ and $\mathbb{Z}/l\mathbb{Z}$ by $D(\mathbb{C}^*)\#\mathbb{Z}/l\mathbb{Z}$.

For a parameter $\kappa = (\kappa_1, \ldots, \kappa_{l-1}) \in \mathbb{C}^{l-1}$, we define a Dunkl operator ∂_κ by

$$\partial_\kappa = \frac{d}{dz} + \frac{l}{z}\sum_{i=0}^{l-1}\kappa_i \mathbf{e}_i$$

where we regard $\kappa_0 = 0$ and let $\mathbf{e}_i = (1/l)\sum_{j=0}^{l-1}\zeta^{ij}\gamma^j$ be an idempotent of $\mathbb{C}(\mathbb{Z}/l\mathbb{Z})$ for $i = 0, 1, \ldots, l-1$.

Definition 4.1 ([EG]). (1) *The rational Cherednik algebra $H_\kappa = H_\kappa(\mathbb{Z}/l\mathbb{Z})$ is the subalgebra of $D(\mathbb{C}^*)\#\mathbb{Z}/l\mathbb{Z}$ generated by z, ∂_κ and γ.*

(2) *The spherical subalgebra of H_κ is the algebra $\mathbf{e}_0 H_\kappa \mathbf{e}_0$.*

The following proposition is analogue of the triangular decomposition of semisimple Lie algebras.

Proposition 4.2 ([EG])**.** *We have the following isomorphisms as $\mathbb{C}$-linear spaces*

$$H_\kappa \simeq \mathbb{C}[z] \otimes_{\mathbb{C}} \mathbb{C}(\mathbb{Z}/l\mathbb{Z}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_\kappa], \quad \textit{and} \quad \mathbf{e}_0 H_\kappa \mathbf{e}_0 \simeq \mathbb{C}[z, \partial_\kappa]^{\mathbb{Z}/l\mathbb{Z}} = \mathbb{C}[z^l, z\partial_\kappa, \partial_\kappa^l].$$

By [BK] together with [Ho], we have the following isomorphism of algebras (Remark 3.10)

$$\begin{aligned} \mathbf{e}_0 H_\kappa \mathbf{e}_0 &\longrightarrow A_c, \\ \mathbf{e}_0 z^l \mathbf{e}_0 &\mapsto \hbar^{-l/2} x_1 \cdots x_l, \\ \mathbf{e}_0 \partial_\kappa^l \mathbf{e}_0 &\mapsto \hbar^{-l/2} y_1 \cdots y_l, \\ \mathbf{e}_0 z\partial_\kappa \mathbf{e}_0 &\mapsto \hbar^{-1} x_1 y_1. \end{aligned} \tag{4}$$

where

$$c = c(\kappa) = (c_i)_{i=0,1,\ldots,l-1}, \qquad c_i = \kappa_i - \kappa_{i+1} - 1/l + \delta_{i,0}. \tag{5}$$

Remark 4.3. *In* [Ku]*, we use parameters $\lambda_i = -c_i$ for quantum Hamiltonian reduction.*

Lemma 4.4 (cf. [Ku], Proposition 4.4)**.** *Assume $c_i + c_{i+1} + \cdots + c_{j-1} \neq 0$ for $0 < i < j \leq l$. Then, the rational Cherednik algebra H_κ is Morita equivalent to its spherical subalgebra $\mathbf{e}_0 H_\kappa \mathbf{e}_0 \simeq A_c$, i.e. we have an equivalence of categories*

$$\begin{aligned} H_\kappa\text{-mod} &\longrightarrow (\mathbf{e}_0 H_\kappa \mathbf{e}_0)\text{-mod}, \\ M &\mapsto \mathbf{e}_0 M. \end{aligned}$$

In the present paper, we assume the assumption of Lemma 4.4 holds.

The category $\mathcal{O}(H_\kappa)$ is the subcategory of H_κ-mod such that the Dunkl operator ∂_κ acts locally nilpotently on a module $M \in \mathcal{O}(H_\kappa)$.

Consider an irreducible $\mathbb{C}(\mathbb{Z}/l\mathbb{Z})$-module $\mathbb{C}\mathbf{e}_i$ for $i = 0, \ldots, l$. We regard $\mathbb{C}\mathbf{e}_i$ as a $\mathbb{C}[\partial_\kappa]\#\mathbb{Z}/l\mathbb{Z}$-module by $\partial_\kappa \mathbf{e}_i = 0$. We define an H_κ-module

$$\mathbf{\Delta}_\kappa(i) = H_\kappa \otimes_{\mathbb{C}[\partial_\kappa]\#\mathbb{Z}/l\mathbb{Z}} \mathbb{C}\mathbf{e}_i$$

called a standard module. By Proposition 4.2, we have

$$\mathbf{\Delta}_\kappa(i) = \mathbb{C}[z]\mathbf{e}_i \tag{6}$$

as a $\mathbb{C}$-linear space.

By the equivalence of Lemma 4.4, we have a subcategory $\mathcal{O}(A_c)$ of A_c-mod which is equivalent to the category $\mathcal{O}(H_\kappa)$. We call $\mathcal{O}(A_c)$ the category $\mathcal{O}$ of A_c. For $i = 1, \ldots, l$, an A_c-module

$$\Delta_c(i) = \mathbf{e}_0 \mathbf{\Delta}_\kappa(i)$$

where c is given by (5) and we regard $\mathbf{e}_l = \mathbf{e}_0$. The module $\Delta_c(i)$ is a standard module for $\mathcal{O}(A_c)$.

The following proposition is fundamental and well-known facts about the category $\mathcal{O}(A_c)$ and modules $\Delta_c(i)$ ($i = 1, \ldots, l$).

Proposition 4.5 ([GGOR])**.** *We have the following fundamental facts about the standard modules $\Delta_c(i)$:*

(1) *For $i = 1, \ldots, l$, the standard module $\Delta_c(i)$ has a unique irreducible quotient $L_c(i)$.*
(2) *The irreducible modules $L_c(i)$ ($i = 1, \ldots, l$) are mutually non-isomorphic.*
(3) *Any simple object in the category $\mathcal{O}(A_c)$ is isomorphic to $L_c(i)$ for some $i = 1, \ldots, l$.*

Remark 4.6. *Originally* [GGOR] *considered the category $\mathcal{O}(H_\kappa)$ of the rational Cherednik algebra H_κ, not one of its spherical subalgebra $\mathbf{e}_0 H_\kappa \mathbf{e}_0 \simeq A_c$.*

By (6), we have

(7) $$\Delta_c(i) = \mathbf{e}_0 \mathbf{\Delta}_\kappa(i) = \mathbf{e}_0 \mathbb{C}[z^l] z^{l-i} \mathbf{e}_i = \mathbb{C}[\hbar^{-l/2} x_1 \cdots x_l] e_i$$

as a $\mathbb{C}$-linear space where we denote $\mathbf{e}_0 z^{l-i} \mathbf{e}_i$ by e_i.

In A_c, we have

$$[\hbar^{-1} x_{\eta_1} y_{\eta_1}, \hbar^{-l/2} x_1 \cdots x_l] = \hbar^{-l/2} x_1 \cdots x_l,$$
$$[\hbar^{-1} x_{\eta_1} y_{\eta_1}, \hbar^{-l/2} y_1 \cdots y_l] = -\hbar^{-l/2} y_1 \cdots y_l.$$

For $i = 1, \ldots, l$, the operator $\hbar^{-1} x_{\eta_1} y_{\eta_1}$ acts semisimply on the standard module $\Delta_c(\eta_i)$, i.e. $\Delta_c(\eta_i)$ is a direct sum of eigenspaces with respect to the action of $\hbar^{-1} x_{\eta_1} y_{\eta_1}$. In fact, by direct calculation we have

$$\Delta_c(\eta_i) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} (\hbar^{-l/2} x_1 \cdots x_l)^m e_{\eta_i},$$

and

(8)
$$(\hbar^{-1} x_{\eta_1} y_{\eta_1}) \circ (\hbar^{-l/2} x_1 \cdots x_l)^m e_{\eta_i} = (m + \tilde{c}_{\eta_1} + \tilde{c}_{\eta_2} + \cdots + \tilde{c}_{\eta_{i-1}})(\hbar^{-l/2} x_1 \cdots x_l)^m e_{\eta_i}.$$

Lemma 4.7. *We have*

$$\Delta_c(\eta_i) = A_c / (A_c(\hbar^{-1} x_{\eta_i} y_{\eta_i}) + A_c(\hbar^{-l/2} y_1 \cdots y_l))$$

for $i = 1, \ldots, l$.

Proof. The standard module $\Delta_c(\eta_i)$ is cyclic with cyclic vector e_{η_i}. By (8), we have $\hbar^{-1} x_{\eta_i} y_{\eta_i} e_{\eta_i} = 0$. Thus, we have the following surjective homomorphism of A_c-modules

$$A_c / (A_c(\hbar^{-1} x_{\eta_i} y_{\eta_i}) + A_c(\hbar^{-l/2} y_1 \cdots y_l)) \twoheadrightarrow \Delta_c(\eta_i),$$
$$f \mapsto f e_{\eta_i}.$$

By Proposition 4.2 and (7), this homomorphism is an isomorphism. $\square$

5. Microlocal construction of modules

5.1. **Construction of the standard modules.** In this section, we introduce the $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_c^\Delta(\eta_i)$ supported on a Lagrangian subvariety $D_i \cup D_{i+1} \cup \cdots \cup D_l$. Moreover, we show that the $\mathcal{M}_c^\Delta(\eta_i)$ is a counterpart of the standard module $\Delta_c(\eta_i)$ of A_c through the equivalence of Theorem 3.9.

Definition 5.1. *For $1 \leq i < i' \leq l$ and a parameter $\lambda = (\lambda_j)_{j=i+1,\ldots,i'} \in \mathbb{C}^{i'-i}$, we call λ admissible when it satisfies $\lambda_j - \lambda_{j+1} - \tilde{c}_j \in \mathbb{Z}$ for $j = i, \ldots, i'-1$ where we regard $\lambda_i = 0$.*

Definition 5.2. *For $i = 1, \ldots, l$, we take an admissible parameter $\lambda = (\lambda_j)_{j=i+1,\ldots,l}$. We define an $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_{c,\lambda}(\eta_i)$ by glueing local sheaves as follows:*

$$\begin{aligned}
\mathcal{M}_{c,\lambda}(\eta_i)|_{X_i} &= \widetilde{\mathscr{A}_{c,i}} / \widetilde{\mathscr{A}_{c,i}} g_i, \\
\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} &= \widetilde{\mathscr{A}_{c,j}} / \widetilde{\mathscr{A}_{c,j}} (f_j \circ g_j - \hbar \lambda_j) \\
&= \widetilde{\mathscr{A}_{c,j}} / \widetilde{\mathscr{A}_{c,j}} (x_{\eta_j} y_{\eta_j} - \hbar \lambda_j) \qquad (\textit{for } j = i+1, \ldots, l), \\
\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} &= 0 \qquad (\textit{for } j = 1, \ldots, i-1).
\end{aligned}$$

Its glueing is given by

(9) $$u_j = f_j^{\lambda_j - \lambda_{j+1} - \tilde{c}_j} u_{j+1} \quad \textit{on } X_j \cap X_{j+1}$$

where u_j *is the image of the constant section* $1 \in \mathscr{A}_{c,j}$ *in* $\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j}$ *for* $j = i, \ldots, l$.

Note that we have

$$\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}, \qquad u_j \mapsto v_{\lambda_j} \tag{10}$$

under the isomorphism $\widetilde{\mathscr{A}_{c,j}} \simeq \mathscr{W}_{T^*\mathbb{C}^1}$.

Lemma 5.3. *The module* $\mathcal{M}_{c,\lambda}(\eta_i)$ *is a well-defined good* $\widetilde{\mathscr{A}_c}$*-module supported on the Lagrangian subvariety* $D_i \cup D_{i+1} \cup \cdots \cup D_l$.

Proof. Set $\mathscr{N}_1 = \mathcal{M}_{c,\lambda}(\eta_i)|_{X_j}$ and $\mathscr{N}_2 = \mathcal{M}_{c,\lambda}(\eta_i|_{X_{j+1}})$. By (3), we have

$$f_{j+1} \circ g_{j+1} - f_j \circ g_j + \hbar \tilde{c}_j = 0$$

on $X_j \cap X_{j+1}$. Thus, we have

$$\begin{aligned}\mathscr{N}_2|_{X_j \cap X_{j+1}} &= \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}} \,/\, \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}}(f_{j+1} \circ g_{j+1} - \hbar\lambda_{j+1}), \\ &= \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}} \,/\, \widetilde{\mathscr{A}_c}|_{X_j \cap X_{j+1}}(f_j \circ g_j - \hbar(\lambda_{j+1} + \tilde{c}_j)).\end{aligned}$$

Since λ is admissible, by Lemma 3.6, $\mathscr{N}_1|_{X_j \cap X_{j+1}}$ is isomorphic to $\mathscr{N}_2|_{X_j \cap X_{j+1}}$ by $u_j \mapsto f_j^{\lambda_j - \lambda_{j+1} - \tilde{c}_j} u_{j+1}$, and consequently $\mathcal{M}_{c,\lambda}(\eta_i)$ is well-defined.

There exists an $\widetilde{\mathscr{A}_c}(0)$-lattice of $\mathcal{M}_{c,\lambda}(\eta_i)$ given by $\mathcal{M}_{c,\lambda}(0)|_{X_j} = \widetilde{\mathscr{A}_c}(0)u_j$ for $j = i, \ldots, l$. Thus, $\mathcal{M}_{c,\lambda}(\eta_i)$ is a good $\widetilde{\mathscr{A}_c}$-module. □

Lemma 5.4. *Fix* $i = 1, \ldots, l$. *Take any admissible parameter* $\lambda = (\lambda_j)_{j=i+1,\ldots,l} \in \mathbb{C}^{l-i}$ *such that* $\lambda \in (\mathbb{C}\backslash\mathbb{Z}_{<0})^{l-i}$ *(resp.* $\lambda \in (\mathbb{C}\backslash\mathbb{Z}_{\geq 0})^{l-i}$*) and* $\lambda' \in \lambda + (\mathbb{Z}_{\geq 0})^{l-i}$ *(resp.* $\lambda' \in \lambda + (\mathbb{Z}_{\leq 0})^{l-i}$*). Then, we have an isomorphism of* $\widetilde{\mathscr{A}_c}$*-modules* $\mathcal{M}_{c,\lambda}(\eta_i) \simeq \mathcal{M}_{c,\lambda'}(\eta_i)$.

Proof. We will prove the case where $\lambda \in (\mathbb{C}\backslash\mathbb{Z}_{<0})^{l-i}$ and $\lambda' \in \lambda + (\mathbb{Z}_{\geq 0})^{l-i}$. It is enough to show that the claim of the lemma holds when there exists $j = i+1, \ldots, l$ such that $\lambda'_j + 1 = \lambda_j$ and $\lambda'_k = \lambda_k$ for $k \neq j$.

By (10), we have

$$\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}, \qquad \mathcal{M}_{c,\lambda'}(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j+1}.$$

Thus, there exists an isomorphism of $\widetilde{\mathscr{A}_{c,j}}$-modules $\mathcal{M}_{c,\lambda}(\eta_i)|_{X_j} \simeq \mathcal{M}_{c,\lambda'}(\eta_i)|_{X_j}$ by Proposition 3.7. For $k \neq j$, we have a trivial isomorphism of $\widetilde{\mathscr{A}_{c,k}}$-modules $\mathcal{M}_{c,\lambda}(\eta_i)|_{X_k} \simeq \mathcal{M}_{c,\lambda'}(\eta_i)|_{X_k}$. These isomorphisms induce an isomorphism of $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_{c,\lambda}(\eta_i) \simeq \mathcal{M}_{c,\lambda'}(\eta_i)$.

The case where $\lambda \in (\mathbb{C}\backslash\mathbb{Z}_{\geq 0})^{l-i}$ and $\lambda' \in \lambda + (\mathbb{Z}_{\leq 0})^{l-i}$ is proved similarly. □

Then, we define the $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(\eta_i)$, $\mathcal{M}_c^{\nabla}(\eta_i)$.

Definition 5.5. *For an admissible parameter* $\lambda \in (\mathbb{C}\backslash\mathbb{Z}_{\geq 0})^{l-i}$, *we denote*

$$\mathcal{M}_c^{\Delta}(\eta_i) = \mathcal{M}_{c,\lambda}(\eta_i).$$

Remark 5.6. *For an admissible parameter* $\lambda \in (\mathbb{C}\backslash\mathbb{Z}_{<0})^{l-i}$, *we denote*

$$\mathcal{M}_c^{\nabla}(\eta_i) = \mathcal{M}_{c,\lambda}(\eta_i).$$

The module $\mathcal{M}_c^{\nabla}(\eta_i)$ *is an* $\widetilde{\mathscr{A}_c}$*-module (conjecturally) corresponding to a costandard module of* A_c.

In the rest of this section, we show that the $\widetilde{\mathscr{A}_c}$-module $\mathcal{M}_c^{\Delta}(\eta_i)$ corresponds to the standard module $\Delta_c(\eta_i)$ via the equivalence of categories Theorem 3.9, i.e. we have $\mathrm{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i)) \simeq \Delta_c(\eta_i)$.

Theorem 5.7. *We have an isomorphism of $\widetilde{\mathscr{A}_c}$-modules $\mathcal{M}_c^{\Delta}(\eta_i) \simeq \widetilde{\mathscr{A}_c} \otimes_{A_c} \Delta_c(\eta_i)$. In other words, we have an isomorphism of A_c-modules,*

$$\mathrm{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i)) \simeq \Delta_c(\eta_i).$$

Proof. By Lemma 4.7, we have

$$\widetilde{\mathscr{A}_c} \otimes_{A_c} \Delta_c(\eta_i) \simeq \widetilde{\mathscr{A}_c}/(\widetilde{\mathscr{A}_c} x_{\eta_i} y_{\eta_i} + \widetilde{\mathscr{A}_c} y_1 \cdots y_l). \tag{11}$$

For $j = 1, \ldots, i$, we have on X_j

$$\begin{aligned} y_1 \cdots y_l &= g_j \circ x_{\eta_1} y_{\eta_1} \circ \cdots \circ x_{\eta_{j-1}} y_{\eta_{j-1}} \\ &= g_j \circ \prod_{k=1}^{j-1} (f_j \circ g_j + \hbar(\tilde{c}_k + \cdots + \tilde{c}_{j-1})). \end{aligned}$$

Since, $\hbar(\tilde{c}_k + \cdots + \tilde{c}_{i-1})$ is nonzero in the field $\mathbb{C}((\hbar))$ for $k = 1, \ldots, i-1$, the isomorphism (11) is reduced on X_j as follows

$$\begin{aligned} &\widetilde{\mathscr{A}_{c,j}} \otimes_{A_c} \Delta_c(\eta_i) \\ &\simeq \widetilde{\mathscr{A}_{c,j}} \Big/ \big\{ \widetilde{\mathscr{A}_{c,j}} (f_j \circ g_j - \hbar(\tilde{c}_j + \ldots \tilde{c}_{i-1})), \\ &\qquad\qquad + \widetilde{\mathscr{A}_{c,j}}\, g_j \circ \textstyle\prod_{k=1}^{j-1} (f_j \circ g_j - \hbar(\tilde{c}_k + \cdots + \tilde{c}_{j-1})) \big\}, \\ &= \widetilde{\mathscr{A}_{c,j}} \Big/ \big\{ \widetilde{\mathscr{A}_{c,j}} (f_j \circ g_j - \hbar(\tilde{c}_j + \cdots + \tilde{c}_{i-1})) + \widetilde{\mathscr{A}_{c,j}} g_j \big\}, \\ &= \begin{cases} \widetilde{\mathscr{A}_{c,i}}/\widetilde{\mathscr{A}_{c,i}} g_i & \text{for } j = i, \\ 0 & \text{for } j = 1, \ldots, i-1. \end{cases} \end{aligned}$$

For $j = i+1, \ldots, l$, we have on X_j,

$$y_1 \cdots y_l = g_j \circ (x_{\eta_1} y_{\eta_1}) \circ \cdots \circ (x_{\eta_i} y_{\eta_i}) \circ \cdots \circ (x_{\eta_{j-1}} y_{\eta_{j-1}})$$

by (2). Since $x_{\eta_k} y_{\eta_k}$ and $x_{\eta_i} y_{\eta_i}$ commute with each other, on X_j we have

$$\begin{aligned} \widetilde{\mathscr{A}_{c,j}} \otimes_{A_c} \Delta_c(\eta_i) &\simeq \widetilde{\mathscr{A}_{c,j}} \Big/ \Big(\widetilde{\mathscr{A}_{c,j}} x_{\eta_i} y_{\eta_i} + \widetilde{\mathscr{A}_{c,j}} g_j \circ \big(\textstyle\prod_{k \neq i} x_{\eta_k} y_{\eta_k}\big) \circ x_{\eta_i} y_{\eta_i} \Big), \\ &= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}} (x_{\eta_j} y_{\eta_j} + \hbar(\tilde{c}_i + \ldots \tilde{c}_{j-1})) \qquad \text{for } j = i+1, \ldots, l. \end{aligned}$$

Note that $\lambda = (\lambda_j)_{j=i+1,\ldots,l} \in \mathbb{C}^{l-i}$ where $\lambda_j = -(\tilde{c}_i + \cdots + \tilde{c}_{j-1})$ is admissible and $\lambda \in (\mathbb{C} \backslash \mathbb{Z}_{\geq 0})^{l-i}$. Thus, the $\widetilde{\mathscr{A}_c}$-module $\widetilde{\mathscr{A}_c} \otimes_{A_c} \Delta_c(\eta_i)$ is clearly isomorphic to $\mathcal{M}_c^{\Delta}(\eta_i)$. □

5.2. **Construction of irreducible modules of $\widetilde{\mathscr{A}_c}$.** In this subsection, we construct modules $\mathcal{L}_c(i)$ of the W-algebra $\widetilde{\mathscr{A}_c}$ for $i = 1, \ldots, l$, and show that they are irreducible modules. Under the equivalence of Theorem 3.9, $\mathrm{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(i))$ is isomorphic to the irreducible module $L_c(i)$ of A_c defined in Section 4.

Fix $i = 1, \ldots, l$. We denote by $\epsilon(i) = i+1, \ldots, l+1$ a unique index such that $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_{\epsilon(i)-1} \in \mathbb{Z}$ and $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_{j-1} \notin \mathbb{Z}$ for any $i < j < \epsilon(i)$.

Definition 5.8. *Fix an admissible parameter $\lambda = (\lambda_{i+1}, \ldots, \lambda_{\epsilon(i)-1}) \in \mathbb{C}^{\epsilon(i)-i-1}$ where we regard $\lambda_{\epsilon(i)} = -1$. We define an $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_{c,\lambda}(\eta_i)$ by glueing local*

sheaves as follows:

$$\begin{aligned}
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_i} &= \widetilde{\mathscr{A}_{c,i}}/\widetilde{\mathscr{A}_{c,i}}g_i,\\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar\lambda_j)\\
&= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(x_{\eta_j}y_{\eta_j} - \hbar\lambda_j) \qquad (\textit{for } j = i+1, \dots, \epsilon(i)-1),\\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_{\epsilon(i)}} &= \widetilde{\mathscr{A}_{c,\epsilon(i)}}/\widetilde{\mathscr{A}_{c,\epsilon(i)}}f_{\epsilon(i)},\\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= 0 \qquad (\textit{for } j = 1, \dots, i-1, \epsilon(i)+1, \dots, l).
\end{aligned}$$

Its glueing is given by

$$u_j = f_j^{\lambda_j - \lambda_{j+1} - \tilde{c}_j} u_{j+1} \quad \textit{on } X_j \cap X_{j+1} \tag{12}$$

where u_j is the image of the constant function $1 \in \widetilde{\mathscr{A}_{c,i}}$ in $\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j}$ for $j = i$, $\dots$, $\epsilon(i)$.

Remark 5.9. *If $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_{j-1}$ for any $j = i+1$, $\dots$, l, we regard $\epsilon(i) = l+1$ and the definition of $\mathcal{L}_{c,\lambda}(\eta_i)$ is given by*

$$\begin{aligned}
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_i} &= \widetilde{\mathscr{A}_{c,i}}/\widetilde{\mathscr{A}_{c,i}}g_i,\\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= \widetilde{\mathscr{A}_{c,j}}/\widetilde{\mathscr{A}_{c,j}}(f_j \circ g_j - \hbar\lambda_j) \qquad (\textit{for } j = i+1, \dots, l),\\
\mathcal{L}_{c,\lambda}(\eta_i)|_{X_j} &= 0 \qquad (\textit{for } j = 1, \dots, i-1).
\end{aligned}$$

Note that we have an isomorphism of $\widetilde{\mathscr{A}_{c,j}}$-modules

$$\mathcal{L}_{c,\lambda}(\eta_i) \simeq \mathscr{M}_{\lambda_j}, \qquad u_j \mapsto v_{\lambda_j} \tag{13}$$

for $j = i+1$, $\dots$, $\epsilon(i)-1$, under the isomorphism $\widetilde{\mathscr{A}_{c,j}} \simeq \mathscr{W}_{T^*\mathbb{C}^1}$.

The following lemmas are proved similarly to Lemma 5.3 and Lemma 5.4 by using (13) instead of (10).

Lemma 5.10. *The module $\mathcal{L}_{c,\lambda}(\eta_i)$ is a well-defined good $\widetilde{\mathscr{A}_c}$-module supported on the Lagrangian subvariety $D_i \cup D_{i+1} \cup \cdots \cup D_{\epsilon(i)}$.*

Proof. The well-definedness is proved similarly to Lemma 5.3.

There exists an $\widetilde{\mathscr{A}_c}(0)$-lattice of $\mathcal{L}_{c,\lambda}(\eta_i)$ given by $\mathcal{L}_{c,\lambda}(\eta_i)(0)|_{X_j} = \widetilde{\mathscr{A}_c}u_j$ for $j = i$, $\dots$, $\epsilon(i)$. Thus, $\mathcal{L}_{c,\lambda}(\eta_i)$ is a good $\widetilde{\mathscr{A}_c}$-module. $\square$

Lemma 5.11. *For any admissible parameters $\lambda, \lambda' \in \mathbb{C}^{\epsilon(i)-i-1}$, we have an isomorphism of $\widetilde{\mathscr{A}_c}$-modules $\mathcal{L}_{c,\lambda}(\eta_i) \simeq \mathcal{L}_{c,\lambda'}(\eta_i)$.*

Proof. Note that we have $\lambda_j \notin \mathbb{Z}$ because λ is admissible and $\tilde{c}_i + \tilde{c}_{i+1} + \cdots + \tilde{c}_j - 1 \notin \mathbb{Z}$ for any $i < j < \epsilon(i)$. Thus this lemma is proved similarly to Lemma 5.4. $\square$

By the above lemma, the $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_{c,\lambda}(\eta_i)$ is independent of the choice of the admissible parameter $\lambda \in \mathbb{C}^{\epsilon(i)-i-1}$.

Definition 5.12. *We denote the $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_{c,\lambda}(\eta_i)$ by $\mathcal{L}_c(\eta_i)$.*

In the rest of this subsection, we show that the $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_c(\eta_i)$ is an irreducible module.

For $i = 1$, $\dots$, l, the good $\widetilde{\mathscr{A}_c}$-module $\mathcal{L}_c(\eta_i)$ is supported on a Lagrangian subvariety $D_i \cup D_{i+1} \cup \cdots \cup D_{\epsilon(i)}$. Thus, $\mathcal{L}_c(\eta_i)$ is a holonomic module. The irreducibility of the holonomic module $\mathcal{L}_c(\eta_i)$ immediately follows from Proposition 3.4 and Proposition 3.8.

Proposition 5.13. *The module $\mathcal{L}_c(\eta_i)$ is an irreducible $\widetilde{\mathscr{A}_c}$-module.*

Proof. Assume there exists a nonzero submodule $\mathcal{N}$ of $\mathcal{L}_c(\eta_i)$. By Proposition 3.4 and Lemma 3.5, we have $\operatorname{Supp}\mathcal{N} = D_j \cup D_{j+1} \cup \cdots \cup D_k$ for some $i \le j \le k \le \epsilon(i)$. Assume $j \neq i$, then $\mathcal{L}_c(\eta_i)|_{X_j}$ is an $\widetilde{\mathscr{A}_{c,j}} \simeq \mathscr{W}_{T^*\mathbb{C}^1}$-module and it has a nontrivial $\mathscr{W}_{T^*\mathbb{C}^1}$-submodule $\mathcal{N}|_{X_j}$ supported on $\{x = 0\}$. On the other hand, by the definition of $\mathcal{L}_c(\eta_i)$, we have $\mathcal{L}_c(\eta_i)|_{X_j} \simeq \mathscr{M}_{\lambda_j}$ and $\lambda_j \notin \mathbb{Z}$. By Proposition 3.8, $\mathcal{L}_c(\eta_i)|_{X_j}$ is an irreducible $\mathscr{W}_{T^*\mathbb{C}^1}$-module, and it contradict the assumption. Thus we have $j = i$. Similarly, we have $k = \epsilon(i)$. Therefore $\mathcal{N} = \mathcal{L}_c(\eta_i)$, and thus, $\mathcal{L}_c(\eta_i)$ is an irreducible $\widetilde{\mathscr{A}_c}$-module. □

Theorem 5.14. *For $i = 1, \ldots, l$, we have*

$$\operatorname{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(\eta_i)) = L_c(\eta_i)$$

Proof. By Proposition 3.8 together with the definitions of $\mathcal{M}_c^{\Delta}(\eta_i)$ and $\mathcal{L}_c(\eta_i)$ (Definition 5.2, Definition 5.8), $\mathcal{L}_c(\eta_i)$ is a quotient of $\mathcal{M}_c^{\Delta}(\eta_i)$. Applying the equivalence of Theorem 3.9, the A_c-module $\operatorname{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(\eta_i))$ is a quotient of $\operatorname{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i)) \simeq \Delta_c(\eta_i)$. Since $\mathcal{L}_c(\eta_i)$ is an irreducible $\widetilde{\mathscr{A}_c}$-module, the A_c-module $\operatorname{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{L}_c(\eta_i))$ is an irreducible quotient of $\Delta_c(\eta_i)$. Therefore, it is isomorphic to $L_c(\eta_i)$. □

Since the full subcategory of regular holonomic $\widetilde{\mathscr{A}_c}$-modules is closed under extensions, we have the following corollary.

Corollary 5.15. *For any module M in $\mathcal{O}(A_c)$, the corresponding $\widetilde{\mathscr{A}_c}$-module $\widetilde{\mathscr{A}_c} \otimes_{A_c} M$ is regular holonomic.*

Next, we discuss the decomposition of the standard modules of $\mathcal{O}(A_c)$ in the Grothendieck group of $\mathcal{O}(A_c)$.

Corollary 5.16. *In the Grothendieck group of $\mathcal{O}(A_c)$, we have*

$$[\Delta_c(\eta_i)] = \sum_{j:\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}} [L_c(\eta_j)].$$

Proof. By Proposition 4.5 (3), we have

$$[\Delta_c(\eta_i)] = \sum_{j=1,\ldots,l} n_j [L_c(\eta_j)]$$

for some $n_j \in \mathbb{Z}_{\ge 0}$. If $\operatorname{Supp}\mathcal{L}_c(\eta_j) \not\subset \operatorname{Supp}\mathcal{M}_c^{\Delta}(\eta_i) = D_i \cup \cdots \cup D_l$, we have $n_j = 0$. Since $\mathcal{M}_c^{\Delta}(\eta_i)$ and $\mathcal{L}_c(\eta_j)$ are (at most) multiplicity-one on D_k, we have

$$\sum_{j:\operatorname{Supp}\mathcal{L}_c(\eta_j) \cap D_k} n_j = 1 \quad \text{for } k = i, i+1, \ldots, l.$$

That is, $[\Delta_c(\eta_i)]$ is multiplicity-free in the Grothendieck group. Since $\mathcal{L}_c(\eta_j)$ is a unique irreducible module whose support is of the form $D_j \cup D_{j+1} \cup \cdots$, we have $n_j = 1$ for $j = i, \ldots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}$ by comparing the support of $\mathcal{M}_c^{\Delta}(\eta_i)$ and $\mathcal{L}_c(\eta_i)$. □

Remark 5.17. *We can also determine the multiplicity $[\Delta_c(\eta_i) : L_c(\eta_j)]$ in the Grothendieck group of $\mathcal{O}(A_c)$ algebraically in this case. The same result of Corollary 5.16 is immediately follows from* [Ku, Lemma 4.3].

Finally, we discuss the subcategory of $\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})$ corresponding to the category $\mathcal{O}(A_c)$. Since a section f of the W-algebra $\widetilde{\mathscr{A}_c}$ is invertible if and only if its

symbol $\sigma_0(f)$ is invertible in $\mathcal{O}_X$, $\hbar^{-1}y_1\cdots y_l$ acts locally nilpotently on an A_c-module M if and only if $\operatorname{Supp}\widetilde{\mathscr{A}_c}\otimes_{A_c} M \subset \bigcup_{i=1}^{l} D_i$. Thus, as mentioned by [Mc, Remark 8.8.2], we have an equivalence of these subcategories:

$$\mathcal{O}(A_c) \simeq \mathrm{Mod}^{good}_{F,\bigcup_{i=1}^{l} D_i}(\widetilde{\mathscr{A}_c})$$

where $\mathrm{Mod}^{good}_{F,\bigcup_{i=1}^{l} D_i}(\widetilde{\mathscr{A}_c})$ is a full subcategory of $\mathrm{Mod}^{good}_{F}(\widetilde{\mathscr{A}_c})$ whose modules are supported on $\bigcup_{i=1}^{l} D_i$. As a corollary of Corollary 5.15, good $\widetilde{\mathscr{A}_c}$-modules with F-action supported on $\bigcup_{i=1}^{l} D_i$ are automatically regular holonomic.

Appendix A. Global sections of modules

We can calculate explicitly the global sections of $\mathcal{M}^{\Delta}(\eta_i)$. Fix $\lambda = (\lambda_j)_{j=i+1,\dots,l} \in \mathbb{C}^{l-i}$ be an admissible parameter. First, the restriction homomorphisms are given explicitly as follows,

$$\begin{aligned}
\mathrm{Res}_1 : \Gamma(X_j, \mathcal{M}_{c,\lambda}(\eta_i)) &\longrightarrow \Gamma(X_j \cap X_{j+1}, \mathcal{M}_{c,\lambda}(\eta_i)),\\
f_j^m u_j &\mapsto f_j^m u_j \qquad (m \in \mathbb{Z}_{\geq 0}),\\
g_j^m u_j &\mapsto C'_{-m,j} f_j^{-m} u_j \qquad (m \in \mathbb{Z}_{>0}),\\
\mathrm{Res}_2 : \Gamma(X_{j+1}, \mathcal{M}_{c,\lambda}(\eta_i)) &\longrightarrow \Gamma(X_j \cap X_{j+1}, \mathcal{M}_{c,\lambda}(\eta_i)),\\
g_{j+1}^m u_{j+1} &\mapsto f_j^{-m+\lambda_{j+1}+\tilde{c}_j-\lambda_j} u_j \qquad (m \in \mathbb{Z}_{\geq 0})\\
f_{j+1}^m u_{j+1} &\mapsto C_{m,j+1} f_j^{m+\lambda_{j+1}+\tilde{c}_j-\lambda_j} u_j, \qquad (m \in \mathbb{Z}_{\geq 0})
\end{aligned}$$

where

$$\begin{aligned}
C_{m,j} &= \hbar^m (m+\lambda_j)(m+\lambda_j-1)\cdots(\lambda_j+1), \qquad (m \in \mathbb{Z}_{\geq 0})\\
C'_{m,j} &= \hbar^{-m} (m+\lambda_j+1)(m+\lambda_j+2)\cdots\lambda_j, \qquad (m \in \mathbb{Z}_{<0})
\end{aligned}$$

are scalar constants. For $j = i, \dots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \notin \mathbb{Z}$, we have $C_{m,j}$, $C'_{m,j} \neq 0$ for all m.

Assume $\lambda = (\lambda_j)_{i+1,\dots,l} \in (\mathbb{C}\backslash\mathbb{Z}_{\geq 0})^{l-i}$. For $j = i, \dots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}$, we have

$$\begin{aligned}
C_{m,j} &\neq 0 \quad (\text{for } m < -\lambda_j, \text{and } j = i, \dots, l-1),\\
C'_{m,j} &\neq 0 \quad (\text{for any } m, \text{and } j = i, \dots, l-1).
\end{aligned} \tag{14}$$

Now, we construct the global sections of $\mathcal{M}_c^{\Delta}(\eta_i)$ explicitly. Fix $i = 1, \dots, l$ and $\lambda_{i+1}, \dots, \lambda_l$ such that $\lambda_j < -\tilde{c}_i - \tilde{c}_{i+1} - \cdots - \tilde{c}_{j-1}$ for all $j = i+1, \dots, l$.

For $j = i, \dots, l$ and $k = j, \dots, l$, set

$$m_{j,k} = -\lambda_k - \tilde{c}_{k-1} - \tilde{c}_{k-2} - \cdots - \tilde{c}_j.$$

Note that we have $m_{j,k} + \lambda_k + \tilde{c}_{k-1} - \lambda_{k-1} = m_{j,k-1}$. For $j = i, \dots, l$ such that $\tilde{c}_i + \cdots + \tilde{c}_{j-1} \in \mathbb{Z}$, take $m \in \mathbb{Z}$ such that $0 \leq m < \tilde{c}_j + \cdots + \tilde{c}_{\epsilon(j)-1}$. [1] Then we define a section

$$v_{j,m} = \begin{cases} (\hbar^{-l/2} f_l)^{m_{j,l}+m} u_l & \text{on } X_l \\ \left(\prod_{k=j'+1}^{l} C_{m_{j,k}+m,k}\right) (\hbar^{-j'+l/2} f_{j'})^{m_{j,j'}+m} u_{j'} & \text{on } X_{j'} \quad (j \leq j' \leq l) \\ 0 & \text{on } X_{j'} \quad (j' \leq j-1) \end{cases}$$

Note that $C_{m_{j,k}+m,k} \neq 0$ by (14), and $v_{j,m}$ is a well-defined global section. Moreover, because $v_{j,m}$ is an F-equivariant section, we can identify it with an F-equivariant homomorphism $\widetilde{\mathscr{A}_c} \ni 1 \mapsto v_{j,m} \in \mathcal{M}_c^{\Delta}(\eta_i)$ in $\mathrm{Hom}_{\mathrm{Mod}_F^{good}(\widetilde{\mathscr{A}_c})}(\widetilde{\mathscr{A}_c}, \mathcal{M}_c^{\Delta}(\eta_i))$.

[1] we regard $\tilde{c}_l = \infty$ here.

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